

On Weil numbers in cyclotomic fields*

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Let p be an odd prime number. It was noticed by Iwasawa that the p -adic behavior of Jacobi sums in $\mathbb{Q}(\zeta_p)$ is linked to Vandiver's Conjecture (see [Iw]). This result has been generalized by various authors for the cyclotomic \mathbb{Z}_p -extensions of abelian fields (see for example [HI], [I1], [B]). In this paper we consider the module of Weil numbers (see §2) for the cyclotomic \mathbb{Z}_p -extension of $\mathbb{Q}(\zeta_p)$, and we get some results quite similar to those for Jacobi sums. In particular we establish a connection between the p -adic behavior of Weil numbers and a weak form of Greenberg Conjecture (see [N2], [BN])

1 Notations

Let p be a fixed odd prime number. For any $n \in \mathbb{N}$ we denote by k_n the p^{n+1} -th cyclotomic field $\mathbb{Q}(\mu_{p^{n+1}})$, where $\mu_{p^{n+1}}$ is the group of p^{n+1} -th roots of unity. We note $\Delta = \text{Gal}(k_0/\mathbb{Q})$, $\Gamma_n = \text{Gal}(k_n/k_0)$ and $G_n = \text{Gal}(k_n/\mathbb{Q})$, so $G_n \simeq \Delta \times \Gamma_n$. Let $\zeta_p \in \mu_p \setminus \{1\}$ and take for any $n \in \mathbb{N}$ $\zeta_{p^{n+1}} \in \mu_{p^{n+1}}$ such that $\forall n \geq 1 \ \zeta_{p^{n+1}}^p = \zeta_{p^n}$. We note $\pi_n = 1 - \zeta_p^{n+1}$

We shall also use the following more or less standard notations:

$k_{n,p}$ the p -completion of k_n ;

$\mathcal{U}_n = 1 + \pi_n \mathbb{Z}_p[\zeta_{p^{n+1}}]$ principal units in $k_{n,p}$;

$\Gamma = \varprojlim \Gamma_n \simeq \mathbb{Z}_p$, γ_0 its topological generator, where $\forall \varepsilon \in \mu_{p^\infty}$, $\gamma_0(\varepsilon) = \varepsilon^{1+p}$;

$\Lambda = \mathbb{Z}_p[[\Gamma]]$ the Iwasawa algebra of the profinite group Γ , $\Lambda \simeq \mathbb{Z}_p[[T]]$ by sending $\gamma_0 - 1$ to T ([W, Theorem 7.1]);

A_n is the Sylow p -subgroup of $\text{Cl}(k_n)$, where $\text{Cl}(k_n)$ is the ideal class group of k_n ;

$X = \varprojlim A_n$ be the projective limit of A_n for the norm maps;

I_n the group of prime-to- p ideals of k_n ;

$k_\infty = \bigcup_{n \in \mathbb{N}} k_n$, $\text{Gal}(k_\infty/k_0) = \Gamma$;

L_n/k_n the maximal abelian unramified p -extension of k_n ; $\text{Gal}(L_n/k_n) \simeq A_n$ by class field theory;

M_n/k_n the maximal abelian p -extension of k_n unramified outside of p ;

$\mathfrak{X}_n = \text{Gal}(M_n/k_n)$;

$L_\infty = \bigcup L_n$; $X \simeq \text{Gal}(L_\infty/K_\infty)$;

$M_\infty = \bigcup M_n$;

$\mathfrak{X}_\infty = \text{Gal}(M_\infty/k_\infty)$.

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Let ψ be a fixed odd character of Δ , different from Teichmüller character ω . We note e_ψ the associated idempotent defined by

$$e_\psi = \frac{1}{|\Delta|} \sum_{\delta \in \Delta} \psi(\delta) \delta^{-1} \in \mathbb{Z}_p[\Delta].$$

Let $\mathcal{M} \in \Lambda$ be the distinguished polynomial of smallest degree such that $\mathcal{M}(T)e_\psi X = \{0\}$. We call it *the minimal polynomial of $e_\psi X$* . It is well known to be prime to $\omega_n = (T+1)^{p^n} - 1$ for any n (cf. [W, §13.6, Theorem 7.10, Theorem 5.11 and Theorem 4.17]).

2 Weil numbers and Jacobi sums.

Fix an n for a moment.

Definition 1 We call Weil module of k_n the module \mathcal{W}_n defined by

$$\begin{aligned} \mathcal{W}_n = \{f \in \text{Hom}_{\mathbb{Z}[G_n]}(I_n, k_n^*) \mid \exists \beta(f) \in \mathbb{Z}[G_n] \text{ such that } \forall \mathfrak{a} \in I_n \\ \mathfrak{a} = (\alpha) \Rightarrow f(\mathfrak{a}) \equiv \alpha^{\beta(f)} \pmod{\mu_{2p^{n+1}}} \} \end{aligned} \quad (1)$$

Observe that $\forall f \in \mathcal{W}_n$, $f(I_n) \subset \mu_{2p^{n+1}}\mathcal{U}_n$.

Definition 2 So we define the module of Weil numbers W_n

$$W_n = \{f(\mathfrak{a}) \mid f \in \mathcal{W}_n, \mathfrak{a} \in I_n\}.$$

Observe that W_n is a submodule of $\mu_{2p^{n+1}}\mathcal{U}_n$.

Let k_n^+ be the maximal totally real subfield of k_n and let G_n^+ stay for $\text{Gal}(k_n^+/\mathbb{Q})$. Let N_n be the norm element in $\mathbb{Z}[G_n]$. Let $N_n^+ \in \mathbb{Z}[G_n]$ be such that its image by the restriction map $\mathbb{Z}[G_n] \rightarrow \mathbb{Z}[G_n^+]$ is $\sum_{\sigma \in G_n^+} \sigma$.

Lemma 1 Let $f \in \mathcal{W}_n$. Then $\beta(f) \in \mathbb{Z}[G_n]$ is unique and

$$\beta(f) \in N_n^+ \mathbb{Z} + \mathbb{Z}[G_n]^-.$$

Proof: Let f be in \mathcal{W}_n and suppose $\beta(f)$ and $\beta'(f)$ verify the required condition.

Let \mathfrak{p} be a split prime ideal in I_n . Let $m \geq 1$ be such that $\mathfrak{p}^m = \alpha \mathcal{O}_{k_n}$. Then

$$f(\mathfrak{p}^m) \equiv \alpha^{\beta(f)} \equiv \alpha^{\beta'(f)} \pmod{\mu_{2p^{n+1}}}.$$

Thus $\mathfrak{p}^{m\beta(f)} = \mathfrak{p}^{m\beta(f')}$, that implies $\mathfrak{p}^{m(\beta(f)-\beta(f'))} = \mathcal{O}_{k_n}$, so $\beta(f) = \beta'(f)$. Furthermore:

$$\beta(f) \in \text{Ann}_{\mathbb{Z}[G_n]}(\mathcal{O}_{k_n}^*/\mu_{2p^{n+1}}) = N_n^+ \mathbb{Z} + \mathbb{Z}[G_n]^-.$$

Proposition 1 *The map $\beta : \mathcal{W}_n \longrightarrow N_n^+ \mathbb{Z} + \mathbb{Z}[G_n]^-$ defined by $f \mapsto \beta(f)$ gives rise to the exact sequence of $\mathbb{Z}[G_n]$ -modules.*

$$0 \longrightarrow \text{Hom}_{\mathbb{Z}[G_n]}(I_n, \mu_{2p^{n+1}}) \longrightarrow \mathcal{W}_n^- \longrightarrow (\text{Ann}_{\mathbb{Z}[G_n]} Cl(k_n))^- \longrightarrow B_n \longrightarrow 0$$

where B_n is a finite abelian elementary 2-group.

Proof:

By the definition of \mathcal{W}_n one has

$$\text{Hom}_{\mathbb{Z}[G_n]}(I_n, \mu_{2p^{n+1}}) = \{f \in \mathcal{W}_n \mid \beta(f) = 0\} = \ker \beta.$$

Note that $f \in \mathcal{W}_n^-$ implies $\beta(f) \in \mathbb{Z}[G_n]^-$. Take $f \in \mathcal{W}_n^-$ and \mathfrak{p} a prime ideal in I_n . Let \mathfrak{p} be a split prime ideal in I_n . Let $m \geq 1$ be such that \mathfrak{p}^m is principal. Then

$$\mathfrak{p}^{m\beta(f)} = f(\mathfrak{p})^m \mathcal{O}_{k_n},$$

that implies

$$\mathfrak{p}^{\beta(f)} = f(\mathfrak{p}) \mathcal{O}_{k_n}.$$

Thus $\beta(f) \in (\text{Ann}_{\mathbb{Z}[G_n]} Cl(k_n))^-$.

Let β be in $(\text{Ann}_{\mathbb{Z}[G_n]} Cl(k_n))^-$ and \mathfrak{p} a prime ideal in I_n . Then there exists $\gamma \in k_n^*$ such that $\mathfrak{p}^\beta = \gamma \mathcal{O}_{k_n}$. Let $\bar{\gamma}$ be the complex conjugate of γ . Then $\bar{\gamma} = \gamma^{-1} \varepsilon$ for some $\varepsilon \in \mathcal{O}_{k_n}^*$. Thus $\varepsilon = \gamma \bar{\gamma}$, i.e. ε is a real unit. Consider $\gamma_1 = \varepsilon^{-1} \gamma^2$. One has: $\gamma_1^{-1} = \bar{\gamma}_1$ and

$$\mathfrak{p}^{2\beta} = \gamma^2 \mathcal{O}_{k_n} = \varepsilon^{-1} \gamma^2 \mathcal{O}_{k_n} = \gamma_1 \mathcal{O}_{k_n}.$$

Let $\gamma_2 \in k_n^*$ such that $\mathfrak{p}^{2\beta} = \gamma_1 \mathcal{O}_{k_n} = \gamma_2 \mathcal{O}_{k_n}$ and $\bar{\gamma}_2 = \gamma_2^{-1}$. Then $\gamma_1 = \gamma_2 \eta$ for some $\eta \in \mathcal{O}_{k_n}^*$, $\gamma_1^{-1} = \gamma_2^{-1} \bar{\eta}$. That implies $\eta \bar{\eta} = 1$, i.e. η is a root of unity.

Now one can choose, for any $\mathfrak{p} \in I_n$, $\gamma_{\mathfrak{p}} \in k_n^*$ such that $\mathfrak{p}^{2\beta} = \gamma_{\mathfrak{p}} \mathcal{O}_{k_n}$, $\bar{\gamma}_{\mathfrak{p}} = \gamma_{\mathfrak{p}}^{-1}$ and $\gamma_{\mathfrak{p}^\sigma} = \gamma_{\mathfrak{p}}^\sigma \forall \sigma \in G_n$. We set:

$$f(\mathfrak{p}) = \gamma_{\mathfrak{p}}$$

and one can verify that $f \in \mathcal{W}_n^-$ and $\beta(f) = 2\beta$. Thus

$$2(\text{Ann}_{\mathbb{Z}[G_n]} Cl(k_n))^- \subset \beta(\mathcal{W}_n^-) \subset (\text{Ann}_{\mathbb{Z}[G_n]} Cl(k_n))^-,$$

that completes the proof. \square

Let $l \neq p$ be a prime number. Let \mathfrak{l} be the prime ideal of k_n above l and $q = |\mathcal{O}_{k_n}/\mathfrak{l}|$. Fix a primitive l -th root of unity ζ_l . The Gauss sum $\tau_n(\mathfrak{l})$ associated to \mathfrak{l} is defined by

$$\tau_n(\mathfrak{l}) = - \sum_{a \in \mathbb{F}_q} \chi_{\mathfrak{l}}(a) \zeta_l^{\text{Tr}_{\mathbb{F}_q/\mathbb{F}_l}(a)}$$

where $\chi_{\mathfrak{l}}$ is a character on \mathbb{F}_q^* of order p^{n+1} defined by

$$\chi_{\mathfrak{l}}(x) \equiv x^{-\frac{q-1}{p^{n+1}}} \pmod{\mathfrak{l}}.$$

One can show that $\forall \delta \in G_n$ on has $\tau_n(\mathfrak{l}^\delta) = \tau_n(\mathfrak{l})^\delta$ (see [W, §6.1]. So we have a well defined morphism of $\mathbb{Z}[G_n]$ -modules

$$\tau_n : I_n \longrightarrow \Omega(\zeta_{p^{n+1}})^*,$$

where Ω is the compositum of all the $\mathbb{Q}(\zeta_m)$, m prime to p .

Definition 3 The Jacobi module \mathcal{J}_n associated to k_n is defined by

$$\mathcal{J}_n = \mathbb{Z}[G_n]\tau_n \cap \text{Hom}_{\mathbb{Z}[G_n]}(I_n, k_n^*),$$

and the module of Jacobi sums J_n is defined by

$$J_n = \{f(\mathfrak{a}) \mid f \in \mathcal{J}_n, \mathfrak{a} \in I_n\}.$$

Let us denote by σ_a the image of $a \in \mathbb{Z}$, prime to p , via the standard isomorphism $(\mathbb{Z}/p^{n+1}\mathbb{Z})^* \simeq G_n$. Let

$$\theta_n = \frac{1}{p^{n+1}} \sum_{a=1, (a,p)=1}^{p^{n+1}} a\sigma_a^{-1}$$

be the Stickelberger element of k_n . Set

$$\mathcal{S}'_n = \sum_{(t,p)=1} \mathbb{Z}[G_n](t - \sigma_t).$$

Definition 4 The Stickelberger ideal of k_n is defined by

$$\mathcal{S}_n = \mathcal{S}'_n \theta_n,$$

(see [W, Lemma 6.9]).

Theorem 1 (Stickelberger's Theorem [S, Theorem 3.1]) Let \mathfrak{p} be a prime ideal in I_n , and $\beta \in \mathcal{S}'_n$. Then $\tau(\mathfrak{p})^\beta \in k_n^*$, $\beta\theta_n \in \mathbb{Z}[G_n]$ and

$$\tau_n(\mathfrak{p})^\beta \mathcal{O}_{k_n} = \mathfrak{p}^{\beta\theta_n}.$$

Moreover, $\tau_n^\beta(\mathfrak{p}) \in \mathcal{U}_n$.

Lemma 2

$$\mathcal{S}'_n = \{\beta \in \mathbb{Z}[G_n] \mid \tau_n^\beta \in \mathcal{J}_n\}.$$

Proof: The inclusion $\mathcal{S}'_n \subset \{\beta \in \mathbb{Z}[G_n] \mid \tau_n^\beta \in \mathcal{J}_n\}$ is obvious by Stickelberger's theorem.

To prove the inverse inclusion it suffices to show that $\tau_n^\beta \in \mathcal{J}_n$ implies $\beta\theta_n \in \mathbb{Z}[G_n]$.

Let $\mathfrak{p} \in I_n$ be a split prime ideal and $\tilde{\mathfrak{p}}$ the unique prime ideal of $\mathbb{Z}[\zeta_{p^{n+1}}, \zeta_l]$ above \mathfrak{p} , where $l = \mathfrak{p} \cap \mathbb{Q}$, $l \equiv 1 \pmod{p^{n+1}}$. Then

$$\tau_n(\mathfrak{p})\mathbb{Z}[\zeta_{p^{n+1}}, \zeta_l] = \tilde{\mathfrak{p}}^{(l-1)\theta_n}.$$

Thus

$$\tau_n(\mathfrak{p})^\beta \mathbb{Z}[\zeta_{p^{n+1}}, \zeta_l] = \tilde{\mathfrak{p}}^{(l-1)\beta\theta_n}.$$

On the other hand,

$$\tau_n(\mathfrak{p})^\beta \mathcal{O}_{k_n} = \mathfrak{p}^z$$

for some $z \in \mathbb{Z}[G_n]$, that implies

$$\tilde{\mathfrak{p}}^{(l-1)z} = \tilde{\mathfrak{p}}^{(l-1)\theta_n\beta}$$

and since $l \equiv 1 \pmod{p^{n+1}}$, one has $(l-1)z = (l-1)\theta_n\beta$. Thus $z = \theta_n\beta$ that implies $\beta\theta_n \in \mathbb{Z}[G_n]$. \square

Proposition 2

- (1) $\mathcal{J}_n \subset \mathcal{W}_n$
- (2) $\mathcal{J}_n \simeq \mathcal{S}_n$.

Proof:

(1) Using the lemma 2 one can easily verify that

$$\mathcal{J}_n = \{\tau_n^\delta \mid \delta \in \mathcal{S}'_n\}.$$

Then for any $f \in \mathcal{J}_n$ there exists $\delta \in \mathcal{S}'_n$ such that $f = \tau_n^\delta$.

Let $f \in \mathcal{J}_n$ and let $\mathfrak{a} \in I_n$ be a principal ideal, $\mathfrak{a} = \alpha \mathcal{O}_{k_n}$. Then by the Stickelberger Theorem one has

$$f(\mathfrak{a}) = \tau_n^\delta(\mathfrak{a}) = \varepsilon \alpha^{\delta \theta_n}$$

for some unit ε . But

$$\tau_n(\mathfrak{a}) \overline{\tau_n(\mathfrak{a})} = N_n(\mathfrak{a}) = N_n(\alpha),$$

so $\varepsilon \in \mu_{2p^{n+1}}$. That means $f(\mathfrak{a}) \equiv \alpha^{\delta \theta_n} \pmod{\mu_{2p^{n+1}}}$, i.e. $f \in \mathcal{W}_n$.

(2) As $\mathcal{J}_n \subset \mathcal{W}_n$ by (1), the map $\beta|_{\mathcal{J}_n}$ is well defined. On the other hand, for any $f \in \mathcal{J}_n$

$$\beta(f) = \beta(\tau_n^\delta) = \delta \theta_n$$

for some $\delta \in \mathcal{S}'_n$. Thus one has a well defined map

$$\begin{array}{ccc} \mathcal{J}_n & \longrightarrow & \mathcal{S}_n \\ \tau_n^\delta & \longmapsto & \delta \theta_n \end{array}$$

This map is obviously surjective (by the definition of \mathcal{S}_n). Its kernel is a submodule of $\text{Hom}_{\mathbb{Z}[G_n]}(I_n, \mu_{p^{n+1}})$ by proposition 1. Let $\delta \in \mathbb{Z}[G_n]$ such that $\delta \theta_n = 0$. Then we have $\sigma_{-1}\delta = \delta$ (σ_{-1} being the complex conjugation in G_n) and $\delta N_n = 0$.

Now let $\delta \in \mathcal{S}'_n$ such that $\delta \theta_n = 0$. Then

$$\tau_n^{\sigma_{-1}\delta} = \tau_n^\delta,$$

and

$$\tau_n^{\delta \sigma_{-1}} = (\tau_n^{\sigma_{-1}})^\delta = \tau_n^{-\delta}$$

as $\theta_n \theta_n^{\sigma_{-1}} = N_n$. Thus $\tau_n^{2\delta} = 1$. Therefore $\tau_n^\delta = 1$ as $\tau_n^\delta \equiv 1 \pmod{\pi_n}$. \square

Lemma 3 *Let $N_{n,n-1}$ be the norm map in the extension k_n/k_{n-1} and $\mathfrak{L} \in I_n$ a prime ideal. Then*

$$N_{n,n-1}(\tau_n(\mathfrak{L})) = \tau_{n-1}(N_{n,n-1}(\mathfrak{L})) \zeta^a l^b,$$

for some $a, b \in \mathbb{Z}$ and some $\zeta \in \mu_{p^{n+1}}$.

For a proof see [I1, Lemma 2].

Remark 1 *The composition $N_{n,n-1} \circ \tau_n$ is well defined because $\text{Gal}(k_n/k_{n-1}) \simeq \text{Gal}(\Omega(\zeta_{p^{n+1}})/\Omega(\zeta_{p^n}))$, $\forall \geq 1$.*

Lemma 4 ([W, Proposition 7.6 (c)]) *The restriction map $\text{Res} : \mathbb{Z}[G_n] \rightarrow \mathbb{Z}[G_{n-1}]$ induces the surjective map*

$$\text{Res} : \mathcal{S}_n^- \longrightarrow \mathcal{S}_{n-1}^-.$$

Proposition 3

$$\forall n \geq 1, N_{n,n-1} \circ \mathcal{J}_n^- \equiv J_{n-1}^- \circ N_{n,n-1} \pmod{\text{Hom}_{\mathbb{Z}[G_n]}(I_n, \mu_{2p^{n+1}})}$$

Proof: Let $f \in \mathcal{J}_n^-$. By the proposition 2 there exists some $\beta \in \mathcal{S}_n^-$ such that $f = \tau_n^\beta$. Let $\mathfrak{L} \in I_n$ be a prime ideal and $\mathfrak{l} = N_{n,n-1}$. Then by the lemmas 3 and 4

$$N_{n,n-1}(\tau_n^\beta(\mathfrak{L})) \equiv \tau_{n-1}^{\text{Res}(\beta)}(\mathfrak{l}) \pmod{\mu_{p^{n+1}}(\mathfrak{L})}.$$

The Proposition follows. \square

3 Annihilators

We recall that ψ is an odd \mathbb{Q}_p -valued character of Δ , irreducible over \mathbb{Q}_p , different from Teichmüller character ω .

Lemma 5 *Let $\mathcal{M} \in \Lambda$ be the minimal polynomial of $e_\psi X$. Then*

$$\varprojlim e_\psi(\text{Ann}_{\mathbb{Z}_p[G_n]} A_n) = \mathcal{M}(T)\Lambda,$$

the projective limit being taken for the restriction maps.

Proof: First we remark that

$$e_\psi \text{Ann}_{\mathbb{Z}_p[G_n]} A_n = \text{Ann}_{\mathbb{Z}_p[\Gamma_n]} e_\psi A_n.$$

We set $A_{n,\psi} = e_\psi A_n$ for simplicity.

Let $\mathcal{M} = (\mathcal{M}_n)_{n \geq 0} \in \Lambda \simeq \varprojlim \mathbb{Z}_p[\Gamma_n]$, the limit being taken with respect for restriction maps. As X^- has no nontrivial finite submodule (see [W, Proposition 13.28]), \mathcal{M}_n annihilates $A_{n,\psi}$, that means $\mathcal{M}_n \mathbb{Z}_p[\Gamma_n] \subset \text{Ann}_{\mathbb{Z}_p[\Gamma]} A_{n,\psi}$. Thus

$$\mathcal{M}(T)\Lambda \subset \varprojlim \text{Ann}_{\mathbb{Z}_p[\Gamma]} A_{n,\psi}.$$

Let $\delta = (\delta_n)_{(n \geq 0)} \in \varprojlim \text{Ann}_{\mathbb{Z}_p[\Gamma]} A_{n,\psi}$. Then for any $n \geq 0$ $\delta_n A_{n,\psi} = \{0\}$. On the other hand,

$$e_\psi X = \varprojlim A_{n,\psi}.$$

Then $\delta e_\psi X = \{0\}$, that implies

$$\delta \in \text{Ann}_\Lambda e_\psi X = \mathcal{M}(T)\Lambda,$$

that completes the proof. \square

Let $\overline{\mathcal{W}}_n = \mathcal{W}_n \otimes \mathbb{Z}_p$ the p -adic adherence of \mathcal{W}_n . The map β of Proposition 1 induces the map

$$\begin{aligned} \overline{\mathcal{W}}_n &\longrightarrow (\text{Ann}_{\mathbb{Z}[G_n]} Cl(k_n))^- \otimes_{\mathbb{Z}} \mathbb{Z}_p = (\text{Ann}_{\mathbb{Z}_p[G_n]} A_n)^- \\ w \otimes a &\longmapsto a\beta(w) \end{aligned}$$

that we shall always note β . Thus we have the short exact sequence

$$0 \longrightarrow \text{Hom}_{\mathbb{Z}[G_n]}(I_n, \mu_{2p^{n+1}}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow \overline{\mathcal{W}}_n \longrightarrow (\text{Ann}_{\mathbb{Z}_p[G_n]} A_n)^- \longrightarrow 0.$$

Applying e_ψ to all the terms of this sequence we get an isomorphism of $\mathbb{Z}_p[G_n]$ -modules

$$e_\psi \overline{\mathcal{W}}_n \simeq e_\psi \text{Ann}_{\mathbb{Z}_p[G_n]} A_n = \text{Ann}_{\mathbb{Z}_p[\Gamma_n]} A_{n,\psi}. \quad (2)$$

Let $z \in \overline{\mathcal{W}}_{n,\psi} = e_\psi \overline{\mathcal{W}}_n$. Then z induces naturally by class field theory (see [Iw, p.455]) a morphism of $\mathbb{Z}_p[\Gamma_n]$ -modules:

$$z : \mathfrak{X}_{n,\psi} \longrightarrow \mathcal{U}_{n,\psi}. \quad (3)$$

Lemma 6 *Let $z \in \overline{\mathcal{W}}_{n,\psi}$ such that $\beta(z) \in \mathbb{Q}_p[\Gamma_n]^*$. Then the kernel of z is $\text{Tor}_{\mathbb{Z}_p[\Gamma_n]} \mathfrak{X}_{n,\psi}$.*

Proof: We have $z(\mathcal{U}_{n,\psi}) = \mathcal{U}_{n,\psi}^{\beta(z)}$. As $\beta \in \mathbb{Q}_p[\Gamma_n]^*$, the quotient $\mathcal{U}_{n,\psi}/\mathcal{U}_{n,\psi}^{\beta(z)}$ is finite. Thus

$$\text{rank}_{\mathbb{Z}_p} z(\mathfrak{X}_{n,\psi}) = \text{rank}_{\mathbb{Z}_p} \mathfrak{X}_{n,\psi} = \text{rank}_{\mathbb{Z}_p} \mathcal{U}_{n,\psi}.$$

Thus $\ker(z : \mathfrak{X}_{n,\psi} \longrightarrow \mathcal{U}_{n,\psi}) = \text{Tor}_{\mathbb{Z}_p[\Gamma_n]} \mathfrak{X}_{n,\psi}$. \square

For any $n \in \mathbb{N}$ let $\mathcal{M}_n \in \mathbb{Z}_p[G_n]$ such that

$$\mathcal{M}_n \equiv \mathcal{M} \pmod{\omega_n}.$$

Then $(\mathcal{M}_n)_{n \geq 0} = \mathcal{M}$ in Λ . Let $w_n \in \overline{\mathcal{W}}_{n,\psi} = e_\psi \overline{\mathcal{W}}_n$ be the element of $\overline{\mathcal{W}}_{n,\psi}$ corresponding to \mathcal{M}_n via the homomorphism (2).

Remark 2 *The Lemma 6 is applicable to w_n , and to the map that consists in multiplication by $e_\psi \theta_n$.*

Lemma 7 *Let $\overline{J}_n = J_n \otimes_{\mathbb{Z}} \mathbb{Z}_p$ be the p -adic adherence of J_n in \mathcal{U}_n and $\overline{W}_n = W_n \otimes_{\mathbb{Z}} \mathbb{Z}_p$ the p -adic adherence of W_n in \mathcal{U}_n . Then*

$$e_\psi \overline{J}_n \subset w_n(\mathfrak{X}_{n,\psi}) \subset e_\psi \overline{W}_n.$$

Proof : One can verify that $\beta(e_\psi \overline{J}_n) = e_\psi \theta_n \mathbb{Z}_p[\Gamma_n]$ (see [W, Chap. 7]). By the Main Conjecture (see [W, §13.6])

$$e_\psi \theta_n \mathbb{Z}_p[\Gamma_n] \subset \mathcal{M}_n \mathbb{Z}_p[\Gamma_n].$$

Set $\widetilde{\mathcal{W}}_{n,\psi}$ the sub- $\mathbb{Z}_p[\Gamma_n]$ -module of $\overline{\mathcal{W}}_{n,\psi}$ generated by w_n . Then $\beta(\widetilde{\mathcal{W}}_{n,\psi}) = \mathcal{M}_n \mathbb{Z}_p[\Gamma_n]$. Thus

$$\beta(e_\psi \overline{J}_n) \subset \beta(\widetilde{\mathcal{W}}_{n,\psi}).$$

As β is an isomorphism, this is equivalent to

$$e_\psi \overline{J}_n \subset \widetilde{\mathcal{W}}_{n,\psi},$$

That implies

$$e_\psi \overline{J}_n \subset w_n(\mathfrak{X}_{n,\psi}). \quad \square$$

Take $(z_n)_{n \geq 1}$, $z_n \in \widetilde{W}_{n,\psi}$ such that $\forall n \geq 1$, $\text{Res}_{n+1,n}\beta(z_{n+1}) = \beta(z_n)$. By the class field theory the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{X}_{n+1,\psi} & \xrightarrow{z_{n+1}} & \mathcal{U}_{n+1,\psi} \\ \downarrow \text{Res}_{n+1,n} & & \downarrow N_{n+1,n} \\ \mathfrak{X}_{n,\psi} & \xrightarrow{z_n} & \mathcal{U}_{n,\psi} \end{array}$$

so the map

$$z_\infty : \mathfrak{X}_{\infty,\psi} \longrightarrow \mathcal{U}_{\infty,\psi} \quad (4)$$

is naturally well defined and

$$z_\infty(\mathfrak{X}_{\infty,\psi}) = \varprojlim z_n(\mathfrak{X}_{n,\psi}) \subseteq \mathcal{U}_{\infty,\psi}$$

Lemma 8 *The kernel of z_∞ is isomorphic to $\alpha(e_{\omega\psi^{-1}}X)$, where $\alpha(e_{\omega\psi^{-1}}X)$ is the Iwasawa adjoint module of $e_{\omega\psi^{-1}}X$.*

Proof : By the definition of z_∞ , $\ker z_\infty = \varprojlim \ker z_n = \varprojlim \text{Tor}_{\mathbb{Z}_p[\Gamma_n]} \mathfrak{X}_{n,\psi}$. But $\varprojlim \text{Tor}_{\mathbb{Z}_p[\Gamma_n]} \mathfrak{X}_{n,\psi} \simeq \alpha(e_{\omega\psi^{-1}}X)$ (see [N1, Proposition 3.1]). \square

Take $z_n = w_n \forall n \geq 1$. Then $e_\psi \overline{J}_\infty \subset w_\infty(\mathfrak{X}_{\infty,\psi})$ by the Lemma 7.

Lemma 9

$$w_\infty(e_\psi \mathcal{U}_\infty) = e_\psi \mathcal{M} \mathcal{U}_\infty$$

Proof : Obvious as $e_\psi \mathcal{U}_\infty$ is free of rank 1. \square

Lemma 10 *The module $W_{\infty,\psi} = \varprojlim \overline{W}_{n,\psi}$ is pseudo-isomorphic to $w_\infty(\mathfrak{X}_{\infty,\psi})$.*

Proof: Let E be the elementary Λ -module such that

$$0 \longrightarrow e_\psi X \longrightarrow E \longrightarrow B \longrightarrow 0,$$

where B is a finite Λ -module. Then $\forall n \gg 0$, $\omega_n B = \{0\}$, and by the snake lemma we obtain the exact sequence

$$0 \longrightarrow B \longrightarrow e_\psi A_n \longrightarrow E/\omega_n E \longrightarrow B \longrightarrow 0.$$

Let $Y_n = \text{Ann}_{\mathbb{Z}_p[\Gamma_n]} E/\omega_n E = \mathcal{M}_n \mathbb{Z}_p[\Gamma_n]$. It is a submodule of $Z_n = \text{Ann}_{\mathbb{Z}_p[\Gamma_n]} e_\psi A_n \simeq e_\psi \overline{\mathcal{W}}_n$, so there exists a submodule $\widehat{\mathcal{W}}_n$ of $e_\psi \overline{\mathcal{W}}_n$ such that $\widehat{\mathcal{W}}_n \simeq Y_n$ as $\mathbb{Z}_p[\Gamma_n]$ -modules. Y_n being monogenous, the same is for $\widehat{\mathcal{W}}_{n,\psi}$, so it is generated by w_n . Thus $\widehat{\mathcal{W}}_n = \widetilde{W}_{n,\psi}$.

There exists $\delta \in \Lambda$, prime to \mathcal{M} , such that $\delta B = \{0\}$. Then $\delta Z_n \subset Y_n$, i.e. $\delta e_\psi \overline{\mathcal{W}}_n \subset \widetilde{\mathcal{W}}_{n,\psi}$. In particular that means

$$\delta e_\psi \overline{W}_n \subset \widetilde{W}_{n,\psi} \subset e_\psi \overline{W}_n, \quad (5)$$

where $\widetilde{W}_n = w_n(\mathfrak{X}_{n,\psi})$. So, taking the projective limit in (5) we obtain

$$\delta e_\psi \overline{W}_\infty \subset w_\infty(e_\psi \mathfrak{X}_\infty) \subset e_\psi \overline{W}_\infty.$$

Thus the quotient module $e_\psi \overline{W}_\infty / w_\infty(e_\psi \mathfrak{X}_\infty)$ is annihilated by two relatively prime polynomials δ and \mathcal{M} , i.e. is finite (see [W, §13.2]). \square

The classical class field theory sequence

$$0 \longrightarrow \overline{\mathcal{O}_{k_n}^* \cap \mathcal{U}_n} \longrightarrow \mathcal{U}_n \longrightarrow \mathfrak{X}_n \longrightarrow A_n \longrightarrow 0$$

gives by taking the ψ -parts the short exact sequence

$$0 \longrightarrow e_\psi \mathcal{U}_n \longrightarrow e_\psi \mathfrak{X}_n \longrightarrow e_\psi A_n \longrightarrow 0, \quad (6)$$

as $\psi \neq \omega$.

Passing to the projective limit in this sequence we obtain the short exact sequence

$$0 \longrightarrow e_\psi \mathcal{U}_\infty \longrightarrow e_\psi \mathfrak{X}_\infty \longrightarrow e_\psi X \longrightarrow 0. \quad (7)$$

Theorem 2

$$\text{char }_{\Lambda}(e_\psi \overline{W}_\infty / e_\psi \overline{J}_\infty) = (\text{char }_{\Lambda} e_\psi X_\infty) / \mathcal{M}(T).$$

Proof: By the Lemma 9, the map w_∞ gives rise to the map

$$\overline{w}_\infty : \frac{e_\psi \mathfrak{X}_\infty}{e_\psi \mathcal{U}_\infty} \longrightarrow \frac{e_\psi \mathcal{U}_\infty}{\mathcal{M} e_\psi \mathcal{U}_\infty}.$$

So in virtue of the sequence (7), we have the map

$$\overline{w}_\infty : e_\psi X \longrightarrow \frac{e_\psi \mathcal{U}_\infty}{\mathcal{M} e_\psi \mathcal{U}_\infty}.$$

The kernel of the map w_∞ being the Λ -torsion module isomorphic to $\alpha(e_{\omega\psi^{-1}} X)$ and $e_\psi \mathcal{U}_\infty$ being a free Λ -module, $\ker(w_\infty) \cap e_\psi \mathcal{U}_\infty = \{0\}$. So $\ker(\overline{w}_\infty) \simeq \alpha(e_{\omega\psi^{-1}} X)$. And we have the following exact sequence

$$0 \longrightarrow \alpha(e_{\omega\psi^{-1}} X) \longrightarrow e_\psi X \longrightarrow e_\psi \mathcal{U}_\infty / \mathcal{M} e_\psi \mathcal{U}_\infty.$$

Let $F = \text{char }_{\Lambda} e_\psi X$ the characteristic polynomial of $e_\psi X$ and $\theta_{\infty,\psi} = (e_\psi \theta_n)_{n \geq 0}$. Then by the Main Conjecture and by the Lemma 7 we obtain the second exact sequence

$$0 \longrightarrow \alpha(e_{\omega\psi^{-1}} X) \longrightarrow e_\psi X \xrightarrow{\times \theta_{\infty,\psi}} e_\psi \mathcal{U}_\infty / F e_\psi \mathcal{U}_\infty$$

These two sequences give two short exact sequences

$$0 \longrightarrow \alpha(e_{\omega\psi^{-1}} X) \longrightarrow e_\psi X \longrightarrow w_\infty(\mathfrak{X}_{\infty,\psi}) / \mathcal{M} e_\psi \mathcal{U}_\infty \longrightarrow 0$$

and

$$0 \longrightarrow \alpha(e_{\omega\psi^{-1}} X) \longrightarrow e_\psi X \longrightarrow e_\psi \overline{J}_\infty / F e_\psi \mathcal{U}_\infty \longrightarrow 0,$$

as $\theta_\infty \mathfrak{X}_{\infty,\psi} = e_\psi \overline{J}_\infty$ (see [B, Lemme 8]). Thus

$$\text{char}_{\Lambda} w_\infty(\mathfrak{X}_{\infty,\psi}) / \mathcal{M} e_\psi \mathcal{U}_\infty = \text{char}_{\Lambda} e_\psi \overline{J}_\infty / F e_\psi \mathcal{U}_\infty. \quad (8)$$

Set $e_\psi \widetilde{W}_\infty = w_\infty(\mathfrak{X}_{\infty,\psi})$. The tautological short exact sequence

$$0 \longrightarrow e_\psi \overline{J}_\infty \longrightarrow e_\psi \widetilde{W}_\infty \longrightarrow e_\psi \widetilde{W}_\infty / e_\psi \overline{J}_\infty \longrightarrow 0$$

gives rise to the short exact sequence

$$0 \longrightarrow \frac{e_\psi \overline{J}_\infty}{F \mathcal{U}_{\infty,\psi}} \longrightarrow \frac{e_\psi \widetilde{W}_\infty}{F \mathcal{U}_{\infty,\psi}} \longrightarrow \frac{e_\psi \widetilde{W}_\infty}{e_\psi \overline{J}_\infty} \longrightarrow 0.$$

Thus

$$\text{char} \frac{e_\psi \overline{J}_\infty}{F \mathcal{U}_{\infty,\psi}} = \text{char} \frac{e_\psi \widetilde{W}_\infty}{F \mathcal{U}_{\infty,\psi}} \left(\text{char} \frac{e_\psi \widetilde{W}_\infty}{e_\psi \overline{J}_\infty} \right)^{-1}. \quad (9)$$

In the same way, the sequence

$$0 \longrightarrow \mathcal{M} e_\psi \mathcal{U}_\infty \longrightarrow e_\psi \widetilde{W}_\infty \longrightarrow e_\psi \widetilde{W}_\infty / \mathcal{M} e_\psi \mathcal{U}_\infty \longrightarrow 0$$

gives rise to the sequence

$$0 \longrightarrow \frac{\mathcal{M} \mathcal{U}_{\infty,\psi}}{F \mathcal{U}_{\infty,\psi}} \longrightarrow \frac{e_\psi \widetilde{W}_\infty}{F \mathcal{U}_{\infty,\psi}} \longrightarrow \frac{e_\psi \widetilde{W}_\infty}{\mathcal{M} \mathcal{U}_{\infty,\psi}} \longrightarrow 0$$

Thus

$$\text{char} \frac{e_\psi \widetilde{W}_\infty}{\mathcal{M} \mathcal{U}_{\infty,\psi}} = \text{char} \frac{e_\psi \widetilde{W}_\infty}{F \mathcal{U}_{\infty,\psi}} \left(\text{char} \frac{\mathcal{M} \mathcal{U}_{\infty,\psi}}{F \mathcal{U}_{\infty,\psi}} \right)^{-1}. \quad (10)$$

Comparing the equalities (8), (9) and (10) we obtain the equality

$$\text{char} \frac{\mathcal{M} \mathcal{U}_{\infty,\psi}}{F \mathcal{U}_{\infty,\psi}} = \text{char} \frac{e_\psi \widetilde{W}_\infty}{e_\psi \overline{J}_\infty}.$$

In virtue of the Lemma 10,

$$\text{char} \frac{e_\psi \widetilde{W}_\infty}{e_\psi \overline{J}_\infty} = \text{char} \frac{e_\psi \overline{W}_\infty}{e_\psi \overline{J}_\infty}.$$

As $\mathcal{U}_{\infty,\psi}$ is free of rank 1,

$$\text{char} \frac{\mathcal{M} \mathcal{U}_{\infty,\psi}}{F \mathcal{U}_{\infty,\psi}} = \frac{F(T)}{\mathcal{M}(T)}.$$

So

$$\text{char} \frac{e_\psi \overline{W}_\infty}{e_\psi \overline{J}_\infty} = \frac{F(T)}{\mathcal{M}(T)}. \quad \square$$

Corollary 1 (cf. [B, Théorème 1])

$$\text{char}_{\Lambda} \frac{\mathcal{U}_{\infty,\psi}}{e_\psi \overline{J}_\infty} = \text{char}_{\Lambda} \alpha(e_{\omega\psi^{-1}} X).$$

Corollary 2 *The module $e_\psi X$ is pseudo-monogenous if and only if the quotient module $e_\psi \overline{W}_\infty / e_\psi \overline{J}_\infty$ is finite.*

By the corollary 1, we see that Greenberg Conjecture implies that

$$\frac{e_\psi \mathcal{U}_\infty}{e_\psi \overline{W}_\infty} \text{ is finite.}$$

So it is natural to ask the following question.

Question: Let p be an odd prime number. Let ψ be an odd character of $\text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$, $\psi \neq \omega$, then, is it true that

$$\frac{e_\psi \mathcal{U}_\infty}{e_\psi \overline{W}_\infty} \text{ is finite ?}$$

Remark 3 *Note that the positive answer to this question is equivalent to*

$$\text{char } e_\psi X = \text{char } \alpha(e_{\omega\psi^{-1}} X) \times \mathcal{M},$$

so it implies weak Greenberg Conjecture (see [BN], [N2]).

References

- [BN] R. Badino, T. Nguyen Quang Do, “Sur les égalités du miroir et certaines formes faibles de la Conjecture de Greenberg”, *Manuscripta Math.* **116** (2005), 323–340.
- [B] T. Beliaeva, ”Unités semi-locales modulo sommes de Gauss”, *J. of Number Theory* **115** (2005) 123–157.
- [HI] Y. Hachimori, H. Ichimura, “Semi-local Units Modulo Gauss Sums”, *Manuscripta Math.* **95** (1998), 377–394.
- [I1] H. Ichimura, “Local Units Modulo Gauss Sums”, *J. Number Theory* **68** (1998), 36–56.
- [Iw] K. Iwasawa, “A note on Jacobi sums”, *Symposia Math.* **XV** (1975), 447–459.
- [N1] T. Nguyen Quang Do, “Sur la \mathbb{Z}_p -torsion de certains modules Galoisiens”, *Ann. Inst. Fourier (Grenoble)* **36** 2 (1986), 27–46.
- [N2] T. Nguyen Quang Do, “Sur la conjecture de Greenberg dans le cas abélien p -décomposé”, *Int. J. Number Theory* **1** vol.2 (2006), 49–64.
- [S] W. Sinnott, “On the Stickelberger ideal and the circular units of an abelian field”, *Invent. Math.* **62** (1980), 181–234.
- [W] L. Washington, “Introduction to Cyclotomic Fields”, G.T.M **83**, Springer, New York 1982.

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